## Applications of $\pi_1(S) \cong \mathbb{Z}$

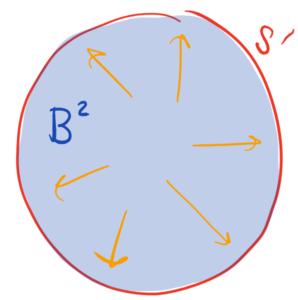
We can use the fact that $\pi_1(S^1) \cong \mathbb{Z}$ to prove several interesting results — all sort of advanced versions of the Intermediate value theorem.

## Fixed points and retractions

Let $B^n$ denote the closed ball of radius 1 in $\mathbb{R}^n$, with boundary the unit sphere $S^{n-1}$

Recall that if $A \subseteq X$, a retraction $r: X \to A$ is a continuous map s.t. $r(a) = a \; \forall \; a \in A$.

**Theorem:** There is no retraction of $B^2$ onto $S^1$.




**Pf:** Suppose $r: B^2 \to S^1$ is a retraction.
Recall from last week that this implies

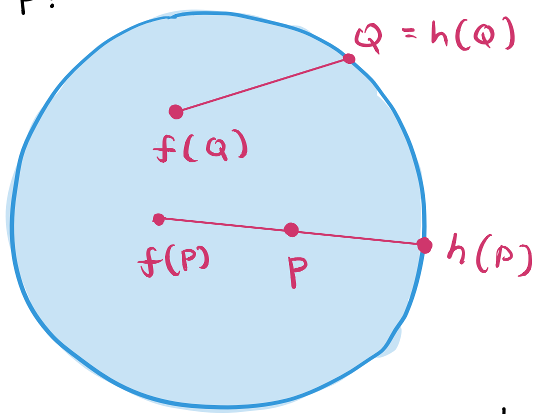
$$r_* : \pi_1(B^2) \to \pi_1(S^1) \text{ is a surjection.}$$

But $B^2$ is convex, so $\pi_1(B^2)$ is trivial, so it can't surject onto $\pi_1(S^1) \cong \mathbb{Z}$. □

**Brouwer Fixed Point theorem:** If $f: B^2 \to B^2$ is continuous, there's some point $x \in B^2$ s.t. $f(x) = x$.

**Pf Sketch #1:** Suppose there is a map $f: B^2 \to B^2$ s.t. $f(x) \neq x \; \forall \; x \in B^2$.

We want to define a map $h: B^2 \to S^1$ that takes $P$ to the point on the boundary $S^1$ that is hit by the ray from $f(P)$ to $P$:




Such a mapping would be a retraction since it would be the identity on $S^1$.

We just need to show $h$ is continuous by finding an explicit formula:

We know $h(P) = P + t(P - f(P))$, where $t$ is the positive number s.t. $\|h(P)\| = 1$. We can then do a tedious calculation to find $t$ and show $h$ is continuous. □

We can give a much slicker proof by using the following lemma:

**Lemma:** Let $h: S^1 \to X$ be continuous. Then the following are equivalent:

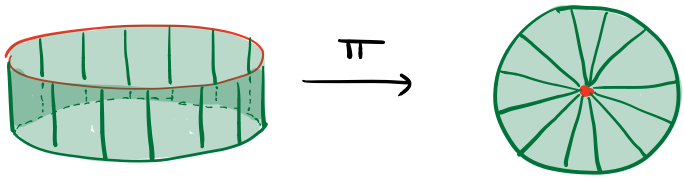
1.) $h$ is nulhomotopic.

2.) $h$ extends to a continuous map $k: B^2 \to X$ (i.e. $k|_{\partial B^2 = S^1} = h$)

3.) $h_*$ is the trivial homomorphism.

**Pf:** 1.) ⇒ 2.): Let $H: S^1 \times I \to X$ be a homotopy between $h$ and a constant map.

Define a map $\pi : S^1 \times I \to B^2$ by $\pi(x,t) = (1-t)x$




$\pi$ is a quotient map (check) that is a homeomorphism away from $S \times \{1\}$, and collapses $S \times \{1\}$ to a point.

Since $H|_{S \times \{1\}}$ is constant, there is a continuous map $k : B^2 \to X$ s.t. $H = k \circ \pi$. So $k(S^1) = H(S^1 \times \{0\}) = h(S^1)$, so $k$ is an extension of $h$.

<u>2.) ⇒ 3.)</u>: Suppose $h$ extends to $k : B^2 \to X$.

Then if $j : S^1 \hookrightarrow B^2$ is the inclusion map, $h = k \circ j$. Thus

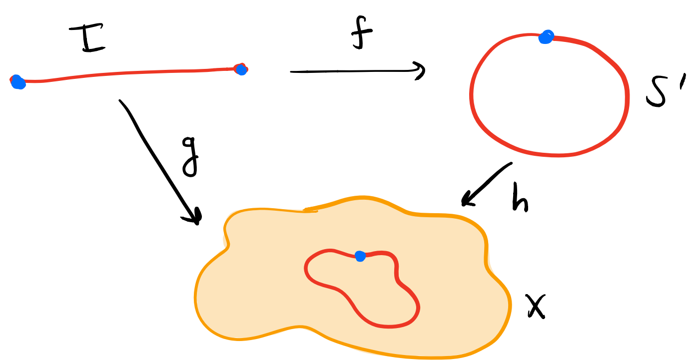
$h_* = k_* \circ j_*$. But $\pi_1(B^2)$ is trivial, so $j_*$ is trivial. Thus, $h_*$ is trivial.

<u>3.) ⇒ 1.)</u>: Assume $h_* : \pi_1(S^1, b_0) \to \pi_1(X, x_0)$ is trivial.

Let $f : I \to S^1$ be the path $f(x) = (\cos(2\pi x), \sin(2\pi x))$

Note that this is also a quotient map

Set $g = h \circ f$.

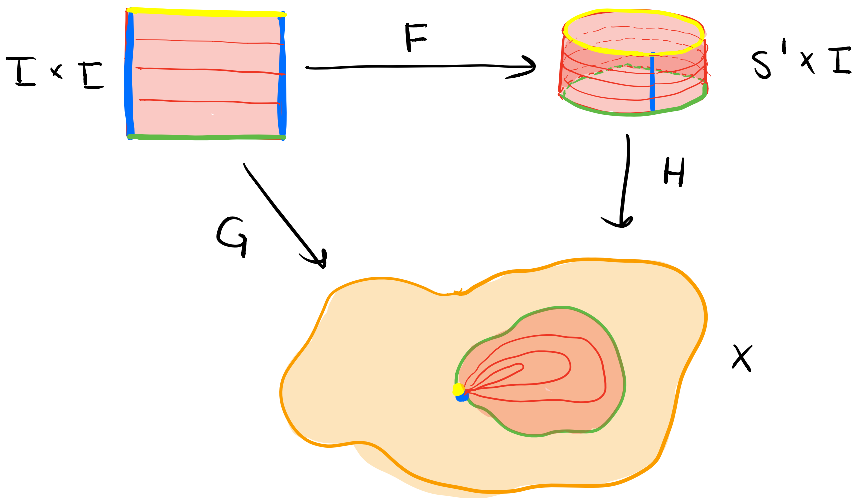


Then $[g]$ is trivial, so there is some path homotopy $G: I \times I \to X$ from $g$ to the constant path at $x_0$.

$F: I \times I \to S^1 \times I$ defined $F(s,t) = (f(s), t)$ is a quotient map and $G(0,t) = G(1,t) = x_0$, so there is a continuous map

$H: S^1 \times I \to X$ s.t. $H \circ F = G$.

So $H$ is a homotopy between $h$ and the constant map. □




**Cor:** The inclusion $j: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ is not nullhomotopic.

**Pf:** $r: \mathbb{R}^2 \setminus \{0\} \to S^1$ defined $r(x) = x/\|x\|$ is a retraction.
Thus, $j_*$ is injective so it's nontrivial. □

**Cor:** The identity $i: S^1 \to S^1$ is not nulhomotopic.

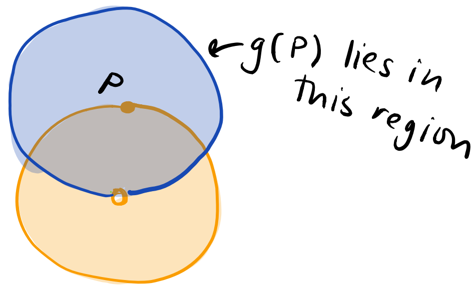
**Pf:** $i_*: \mathbb{Z} \to \mathbb{Z}$ is the identity and thus nontrivial. □

Now we can give a nicer proof of the Brouwer fixed-point thm:

**Proof #2 of BFPT:** Suppose $f: B^2 \to B^2$ has no fixed point.

Then we can define $g: B^2 \to \mathbb{R}^2 \setminus \{0\}$ by $g(P) = P - f(P)$.

For $P \in S^1$, $g(P)$ lies in a convex region around $P$:




Thus, we can use the straight-line homotopy

$H: S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$ defined $H(P, t) = (1-t)P + t\,g(P)$.

(Note: $(1-t)P + t\,g(P) = P - tP + tP - tf(P) = P - tf(P) \neq 0$)

$H$ is a homotopy from $g|_{S^1}$ to the inclusion. Thus $g|_{S^1}$ is <u>not</u> nulhomotopic, which is a contradiction. $\square$